An Introduction to Mapping Class Groups

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Outline of this talk

- Preliminaries: Surfaces, Homeomorphisms and Homotopies
- Dehn Twists
- Mapping Class Groups
- Connection to Braid Groups

Surfaces

Definition

A **surface** is a two-dimensional manifold with or without boundary. Informally, it is a geometrical shape that resembles a deformed plane.

Example

Boundaries of solid objects in $\mathbb{R}^3$, such as a sphere and torus, are the most familiar examples.

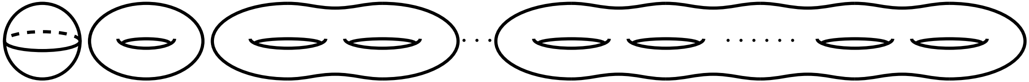


Figure: A list of surfaces without boundary.

Properties of Surfaces

Definition

A **compact** surface is a surface that is also a closed and bounded set.

Definition

Let S be a surface. The **boundary** of S is the collection of points on S minus the set of all interior points of S, i.e. $\partial S = S \setminus \text{int}(S)$. If $\partial S \neq \emptyset$, then S is a **surface with boundary**.

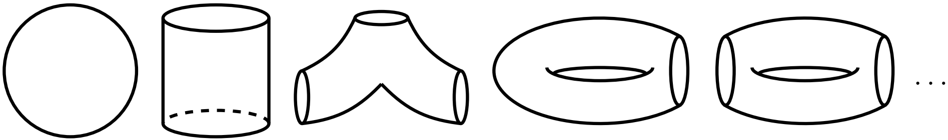


Figure: A list of surfaces with boundary.

Properties of Surfaces

- An **orientable** surface allows a consistent definition of "clockwise" and "counterclockwise." On the other hand, a surface is **non-orientable** if and only if it contains a Möbius band.
- The **genus** g of an orientable surface S is an integer representing the number of handles, or holes, on S.

Example

The surfaces without boundary are listed in ascending order from genus 0 to genus n.

Homeomorphisms

Definition

Let S be a surface. A **homeomorphism** $f : S \to S$ is a continuous bijection with a continuous inverse.

Example

Rotations, reflections and hyperelliptic involutions. Note that the orientation changes under a reflection. Moreover, a special example of a homeomorphism that cannot be realized by rigid motions is a Dehn twist.

Example

Using polar coordinates, define the rotation by angle θ as

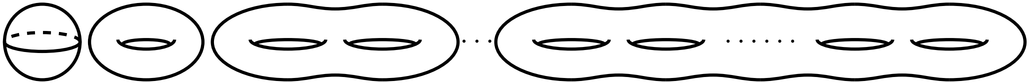
$$\begin{aligned} f_\theta : S^1 &\to S^1 \\ (1, \alpha) &\mapsto (1, \alpha + \theta), \end{aligned}$$

where α is any angle.

Classification of Surfaces

Theorem

Every compact, orientable surface without boundary is homeomorphic to one of the surfaces below.



In other words, compact, orientable surfaces without boundary are homeomorphic if and only if they share the same genus g.

Homotopy Between Homeomorphisms

Definition

Let S be an orientable surface and let $f : S \to S$ and $g : S \to S$ be two homeomorphisms. We call f and g **homotopic** if there exists a continuous map $H : S \times [0,1] \to S$ such that $H_0 = f$ and $H_1 = g$, where $H_t(x) = H(x,t)$.

Lemma

The identity map id_{S^1} is homotopic to the homeomorphism f_θ.

Proof. Consider the map

$$\begin{aligned} H : S^1 \times [0,1] &\to S^1 \\ (\alpha, 0) &\mapsto \alpha \\ (\alpha, 1) &\mapsto \alpha + \theta. \end{aligned}$$

Then $H(\alpha, t) = \alpha + \theta t$, where $H_0 = id_{S^1}$ and $H_1 = f_\theta$. Thus, id_{S^1} is homotopic to f_θ. □

More on Homotopy

- The identity map id_{S^2} is homotopic to the homeomorphism $f_\theta : S^2 \to S^2$.
- Homotopy defines an equivalence relation.

Dehn Twists About an Annulus

Definition

We define a **Dehn twist** on A as follows:

$$\begin{aligned} T_A : A &\longrightarrow A \\ (r, \theta) &\longmapsto (r, \theta - 2\pi r), \end{aligned}$$

where the boundary of A, denoted ∂A, is fixed pointwise.

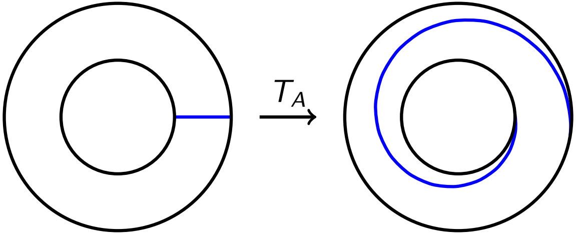


Figure: A Dehn twist on an annulus A.

Dehn Twist About Simple Closed Curves

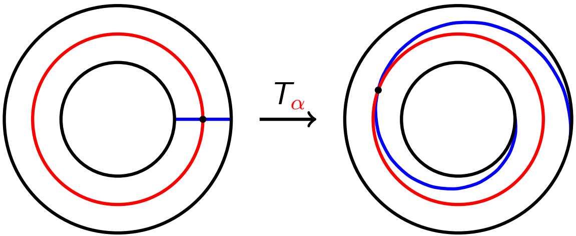


Figure: A Dehn twist about the red simple closed curve α.

We can consider the **core** of an annulus A, which is the set of points when $r = \frac{3}{2}$. Every simple closed curve, i.e. loops without self-intersections, on an orientable surface is the core of some annulus.

Dehn Twists on a Surface

Definition

Let S be an orientable surface with two simple closed curves α and β. Then a **Dehn twist** about α on S is obtained by choosing an annulus A, applying T_α and extending by the identity, i.e. fixing every point in $S \setminus A$. Similarly, a Dehn twist about β on S is obtained by choosing an annulus A, applying T_β and extending by the identity.

Dehn Twists on a Surface

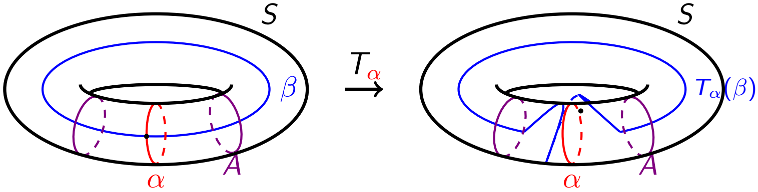


Figure: We realize the simple closed curve α as the core of annulus A.

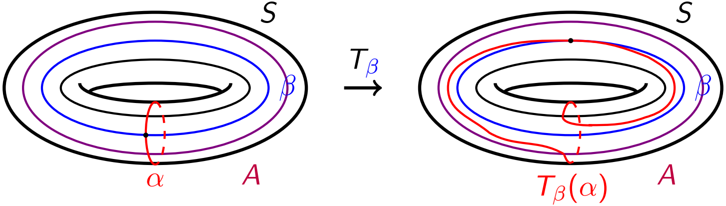


Figure: We realize the simple closed curve β as the core of annulus A.

Remark

In the previous figure, suppose we choose a different annulus A'. Then we will obtain a curve $T'_\alpha(\beta)$ that is homotopic to $T_\alpha(\beta)$. Moreover, we have that T_α and T'_α are homotopic homeomorphisms and, thus, belong to the same mapping class, which we will define in the next section. In particular, any choice of annulus will yield an element of the mapping class of T_α.

Mapping Class Groups

We can consider the set of all homeomorphisms of a surface S, denoted by $\mathrm{Homeo}(S)$. For surfaces with boundary, we only consider the homeomorphisms that fix ∂S pointwise. The set of all homeomorphisms that are homotopic to identity 1_S is denoted by $\mathrm{Homeo}_0(S)$.

Lemma

$\mathrm{Homeo}(S)$ is a group, with $\mathrm{Homeo}_0(S)$ as a normal subgroup.

Proof. Clearly, $\mathrm{Homeo}(S)$ is a group since function composition is associative, $1_S \in \mathrm{Homeo}(S)$ and for any $f \in \mathrm{Homeo}(S)$, its inverse $f^{-1} \in \mathrm{Homeo}(S)$, by definition. Moreover, $\mathrm{Homeo}_0(S)$ is a subgroup of $\mathrm{Homeo}(S)$. Note that $1_S \in \mathrm{Homeo}_0(S)$ since 1_S is homotopic to itself. Let $f, g \in \mathrm{Homeo}_0(S)$. We can define the homotopy $H(x,t) = F(G(x,t),t)$, where $F : S \times [0,1] \to S$ is a homotopy such that $F_0 = 1_S$ and $F_1 = f$ and $G : S \times [0,1] \to S$ is a homotopy such that $G_0 = 1_S$ and $G_1 = g$ for all $x \in S$. Then $H_0(x) = x$ and $H_1(x) = f(g(x))$. So it follows that $\mathrm{Homeo}_0(S)$ is closed under function composition.

Furthermore, let $f, g, h \in \mathrm{Homeo}(S)$ with f and g homotopic. Then $h \circ f$ is homotopic to $h \circ g$ since we can construct the homotopy $K(x,t) = h(H(x,t))$ such that $H : S \times [0,1]$, where $H_0 = f$ and $H_1 = g$. Using this fact, we deduce that for any $f \in \mathrm{Homeo}_0(S)$, also $f^{-1} \in \mathrm{Homeo}_0(S)$ when we notice that $f^{-1} \circ f$ is homotopic to $f^{-1} \circ 1_S$.

Now we show that $\mathrm{Homeo}_0(S) \trianglelefteq \mathrm{Homeo}(S)$. It suffices to show that for any $g \in \mathrm{Homeo}(S)$ and $f \in \mathrm{Homeo}_0(S)$, we have that $gfg^{-1} \in \mathrm{Homeo}_0(S)$. Since $f \in \mathrm{Homeo}_0(S)$, there is $F : S \times [0,1] \to$ such that $F_0 = f$ and $F_1 = 1_S$. Note that for any $0 \leq t \leq 1$,

$$H(x,t) = g(F(g^{-1}(x),t))$$

such that $H_0 = gfg^{-1}$ and $H_1 = 1_S$ is continuous. Therefore, our proof is complete. $\square$

Remark

Definition

The **fundamental group** of a topological space X, denoted $\pi_1(X, x)$, is the group of homotopy classes of x-based loops in X.

Recall that if f is homotopic to g and α is a simple closed curve, then $f(\alpha)$ and $g(\alpha)$ are homotopic curves. The fundamental group of a surface captures the group structure of equivalence classes of simple closed curves on a surface.

Definitions and Elementary Examples

Definition

Let S be an orientable surface. The **mapping class group** of S, denoted by $\mathrm{MCG}(S)$, is the group of homotopy classes of orientation-preserving homeomorphisms of S, i.e. $\mathrm{MCG}(S) = \mathrm{Homeo}_+(S) \ / \ \mathrm{Homeo}_0(S)$.

Elements of the mapping class group are called **mapping classes**.

Example

Recall that id_{S^1} and f_θ belong to the same mapping class in $\mathrm{MCG}(S^1)$. In fact, every homeomorphism of S^1 is homotopic to id_{S^1}, so $\mathrm{MCG}(S^1)$ is trivial.

Example

Similarly, id_{S_2} and f_θ belong to the same mapping class in $\mathrm{MCG}(S^2)$.

Definitions and Elementary Examples

Theorem (Alexander Lemma)

The mapping class group of the closed disk D^2 is trivial.

Proof. Let $f : D^2 \to D^2$ be a homeomorphism and assume that $f(x) = x$ for any $x \in \partial D^2$. We want to show that f is homotopic to id_{D^2}. Consider the map $H : D^2 \times [0,1] \to D^2$, where

$$H(x,t) = \begin{cases} x, & \text{if } 1-t \leq |x| \leq 1 \\ (1-t)f(\frac{x}{1-t}), & \text{if } 0 \leq |x| < 1-t \end{cases}$$

for $t \in [0,1)$. Moreover, define $H(x,1) = id_{D^2}$.
Note that H is continuous, and thus, the homotopy between f and id_{D^2}. Therefore, $\mathrm{MCG}(D^2) = \{id_{D_2}\}$. $\square$

We call the previous proof, the *Alexander trick*.

The Simplest Infinite Order Mapping Class Group

Definition

Let X be a topological space. A **path** in X is a continuous function $f : [0, 1] \to X$.

Definition

A topological space X is **simply-connected** if any loop in the space can be continuously deformed into a single point, i.e. is *contractible*.

The fundamental group of X at each point in the space measures how far X is from simply-connectedness. A **path-connected** space is simply-connected if and only if its fundamental group is trivial.
A surface S is simply-connected if and only if it is *connected* with genus 0.

Example

S^2 is simply-connected.

The Simplest Infinite Order Mapping Class Group

Definition

A **covering** of a space X is another space E together with a map $\Phi : E \to X$ such that for any point $x \in X$, there exists an open neighborhood U of x such that $\Phi^{-1}(U)$ is a disjoint union of open sets in E, each of which is mapped homeomorphically onto U.

Definition

A **universal cover** of X is a covering space that is simply-connected.

If a connected topological space X is simply-connected, then it is its own universal cover.

Example

S^2 is its own universal cover.

Example

$\mathbb{R}$ is the universal cover of S^1. Note that $\mathbb{R}$ is a simply-connected space with the covering map $f : \mathbb{R} \to S^1$ such that $f(t) = e^{2\pi it}$.

The Simplest Infinite Order Mapping Class Group

Example

Let A be an annulus. The universal cover of A is the infinite strip $\tilde{A} \approx \mathbb{R} \times [0,1]$ since A is homeomorphic to $S^1 \times [0,1]$.

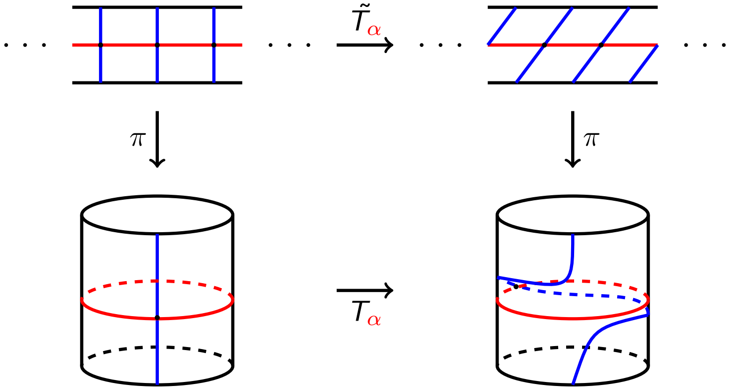


Figure: A preferred lift of a Dehn twist about the red simple closed curve α on A.

The Simplest Infinite Order Mapping Class Group

Theorem

MCG(A) $\approx \mathbb{Z}$.

Proof. We construct a map $\rho : \mathrm{MCG}(A) \to \mathbb{Z}$. Let $f \in \mathrm{MCG}(A)$ and let $\varphi : A \to A$ be any homeomorphism representing the mapping class f. Then φ has a preferred lift $\tilde{\varphi} : \tilde{A} \to \tilde{A}$ such that $\tilde{\varphi}|_{\mathbb{R}\times\{0\}} = id|_{\mathbb{R}\times\{0\}}$. Now let $\tilde{\varphi_1} : \mathbb{R} \to \mathbb{R}$ denote the restriction $\tilde{\varphi}|_{\mathbb{R}\times\{1\}}$. Note that we can canonically identify $\tilde{\varphi_1}$ with $\mathbb{R}$. Next, we define $\rho(f) = \tilde{\varphi_1}(0)$. Notice that any homeomorphism homotopic to identity satisfies $\tilde{\varphi_1}(0) = 0$ since $\tilde{\varphi}|_{\mathbb{R}\times\{1\}} = id|_{\mathbb{R}\times\{1\}}$. For any homeomorphism that is not homotopic to identity, $\tilde{\varphi_1}(0) = n$, where $n \in \mathbb{Z} \setminus \{0\}$. This follows from the fact that ∂A is fixed pointwise and we only consider simple closed curves on A, so there cannot be any intersections of arcs in $\tilde{A}$. Hence, $\rho(f) = \tilde{\varphi_1}(0) \in \mathbb{Z}$. Since compositions of homeomorphisms of A map to compositions of integer translations of $\mathbb{R}$, it is clear that ρ is a homomorphism.

Further, we find that $\ker(\rho)$ is trivial since ∂A is fixed pointwise and any homeomorphism homotopic to identity satisfies $\tilde{\varphi}_1(0) = 0$. Every homeomorphism homotopic to identity lifts to arcs homotopic to the ones shown in the previous example on the left infinite strip. Thus, ρ is injective. And surjectivity follows from the existence of a homeomorphism for each integer translation. $\square$

The Mapping Class Group of the Torus

Theorem

$MCG(T^2) \cong SL_2(\mathbb{Z})$.

Proof. (Idea) We use a similar method as in the previous theorem. Construct a map $\sigma : \mathrm{MCG}(T^2) \to \mathrm{SL}_2(\mathbb{Z})$. Note that $\mathbb{R}^2$ is the universal cover of T^2. Let T_α be a Dehn twist about α on T^2, which can clearly be representative of an element of $\mathrm{MCG}(T^2)$. Then T_α has a preferred lift $\tilde{T}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$.

The Mapping Class Group of the Torus

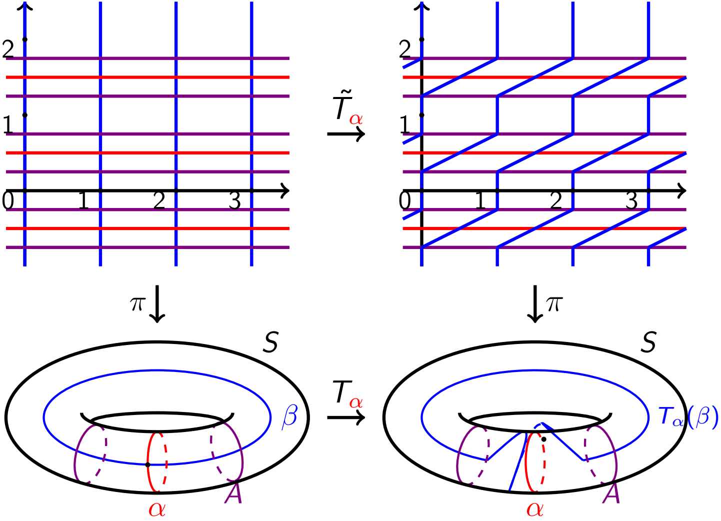


Figure: A preferred lift of a Dehn twist about α on T^2.

The Mapping Class Group of the Torus

Every simple closed curve on a torus can be homotoped to intersect a point and lifts to a line through the origin which also passes through another integer point. In fact, the first such point is (n, m), where $\gcd(n, m) = 1$. Moreover, since there is a bijective correspondence between nontrivial homotopy classes of oriented simple closed curves on T^2 and the primitive elements of Z^2, there must exist some matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $A((n, m)) = (1, 0)$. Notice that $\tilde{T}_\alpha(1, 0) = (1, 0)$ and $\tilde{T}_\alpha(0, 1) = (1, 1)$. Thus, $\tilde{T}_\alpha$ is a linear, orientation-preserving homeomorphism of $\mathbb{R}^2$ preserving $\mathbb{Z}^2$. It follows that $\tilde{T}_\alpha$ is isomorphic to T_α, where $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T_\alpha(1, 0) = (1, 0)$ and $T_\alpha(0, 1) = (1, 1)$. So we can represent a Dehn twist about α as

$$T_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The Mapping Class Group of the Torus

Now let T_β be a Dehn twist about β, which is another representative of a mapping class in $\mathrm{MCG}(T^2)$. Then we deduce that $\tilde{T}_\beta$ is isomorphic to T_β, where $T_\beta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T_\beta(1,0) = (1,-1)$ and $T_\beta(0,1) = (0,1)$. Hence, we can represent a Dehn twist about β as

$$T_\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Recall that $\mathrm{SL}_2(\mathbb{Z})$ is the set of all 2×2 matrices with integer entries and determinant 1. Moreover, it is generated by the matrices T_α and T_β. It turns out that the mapping class group of the torus is generated by the same matrices. □

Connection to Braid Groups

Definition

The **braid group** on n strands, denoted B_n, is the group of equivalence classes of $n-$braids.

Lemma

$B_2 \cong \mathbb{Z} \approx MCG(A)$.

Proof. Note that the braid group B_2 has the presentation

$$\begin{aligned} B_2 &= \langle \sigma_1, \cdots, \sigma_{n-1} \,|\, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \,\&\, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\, 1 \leq i \leq n-2 \rangle \\ &= \langle \sigma_1 \rangle. \end{aligned}$$

Since B_2 is generated by a single element, and thus cyclic, it follows that B_2 is isomorphic to $\mathbb{Z}$. $\square$

Connection to Braid Groups

Now we return to mapping class groups and consider a disk with n punctures D_n. Then define a map $\varphi : B_n \to \text{MCG}(D_n)$. Given a braid, we slide the disk across the braid to obtain a homeomorphism. We can visualize this as follows: each puncture is connected by a string to the boundary of the disk and each mapping homomorphism that permutes two of the punctures can then be seen to be a homotopy of the strings, i.e. a braid. It turns out that φ is indeed an isomorphism.
By $\text{MCG}(D_n)$, we denote the group of mapping classes of homeomorphisms of an n-punctured disk which fix points on the boundary of the circle pointwise, but not necessarily the n punctures.

Theorem

The mapping class group of an $n-$punctured disk $MCG(D_n)$ is isomorphic to the braid group B_n.

Connection to Braid Groups

Definition

A **configuration space** is the set of all possible *ordered* configurations of n particles

$$C_n(\mathbb{R}^2) = \{(p_1, p_2, \cdots, p_n) \in (\mathbb{R}^2)^n \mid p_i \neq p_j \text{ for } i \neq j\},$$

where $p_i \neq p_j$ is the condition that the particles must not collide. We can also consider the set of all possible *unordered* configurations of n particles

$$UC_n(\mathbb{R}^2) = \{\{p_1, p_2, \cdots, p_n\} \subset \mathbb{R}^2 \mid p_i \neq p_j \text{ for } i \neq j\}.$$

Note that the left wall and right wall of a configuration space represent points in $UC_n(\mathbb{R}^2)$. In fact, they can be realized as the same point so we can form the notion of a *loop* in the space.

Theorem

The fundamental group of $UC_n(\mathbb{R}^2)$ is isomorphic to the braid group B_n.

Three Perspectives of Braids

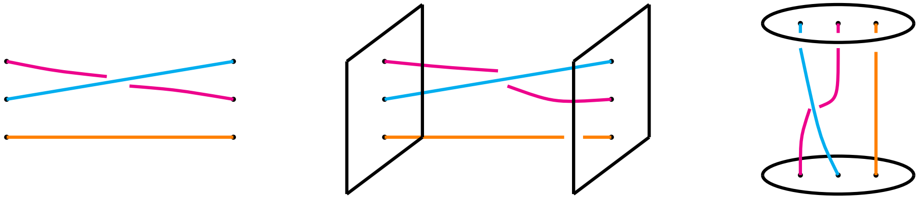


Figure: Three perspectives of B_3: traditional braid, configuration space and 3-punctured disk.

A Word on the Nielsen-Thurston Classification

Using this mapping class group interpretation of braids, each braid can be classified as *periodic*, *reducible* or *pseudo-Anosov*.

We love mapping class groups!

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Thank you for listening!

Any questions?

Special thanks to the Organizers & Kasia Jankiewicz :)